A regular model for bond percolation: the approach towards the threshold

1985 J. Phys. A: Math. Gen. 18 L1149

(http://iopscience.iop.org/0305-4470/18/18/007)

J. Phys. A: Math. Gen. 18 (1985) L1149-L1155. Printed in Great Britain

LETTER TO THE EDITOR

A regular model for bond percolation: the approach towards the threshold

Takashi Nagatani
College of Engineering, Shizuoka University, Hamamatsu 432, Japan

Received 5 August 1985, in final form 16 September 1985

Abstract. A two-dimensional regular model is constructed for bond percolation via a rule of bond deletions on the square lattice. The model describes the approach towards the threshold. The critical bond concentration p_c and the correlation length exponent ν are found. The fractal geometry of the infinite percolating network and the scaling property of the cluster size distribution are studied above the threshold. The infinite percolating network is shown to be reconstructed by the Mandelbrot-Koch curve proposed as the non-random model for the infinite cluster at criticality.

Recently, there has been increasing interest in exact mathematical fractals (Mandelbrot 1982, Vicsek 1983, Given and Mandelbrot 1983, Ben-Avraham and Havlin 1983). The main reason is that the solution of many important equations of physics on these lattices adds to our understanding of the geometric and topological properties that are relevant to modelling the corresponding physical processes. The percolating infinite cluster is one of the most intensively studied random fractals (Deutscher *et al* 1983, Stauffer 1979, 1985, Stanley and Coniglio 1983, Kirkpatrick 1979, Kapitulnik and Deutscher 1984). Various geometrical models have been proposed to imitate the infinite incipient cluster at the percolation threshold, and it is of great interest to understand the effects of these different geometries on the transport properties at the percolation threshold. Three extreme models for the backbone of the infinite cluster have been proposed, i.e. the family of Sierpinski gaskets, the 'links and nodes' model and the 'links-nodes-blobs' model (Coniglio 1982, Aharony *et al* 1984). Mandelbrot (1984a, b) and Mandelbrot and Given (1984) have also presented fractal models for percolation clusters at criticality. The Mandelbrot models possess the geometric and topological properties very close to the infinite cluster at the percolation threshold but do not describe the approach towards the threshold p_c.

In general, every lattice bond has three choices in the bond percolation: it can be empty, with probability $1-p$; it can be part of the infinite network of occupied bonds, with probability pP_∞ (P_∞: the percolation probability); or it can be part of one of the many finite clusters, with probability $p(1-P_\infty)$. Since each s cluster contains exactly s bonds, the probability of any lattice bond belonging to an s cluster is $P_s = sn_s$ (n_s is the number of s clusters divided by the total number of lattice bonds). The sum of all these probabilities equals unity. As the concentration p approaches the threshold p_c, the pair connectedness length ξ diverges, $\xi \sim (p-p_c)^{-\nu}$. The percolation probability P_∞, the cluster size distribution n_s and the typical cluster size s_ξ show the following scaling behaviours (Stauffer 1979, 1985):

$$P_\infty(p) \sim (p-p_c)^\beta \qquad n_s(p) \sim s^{-\tau} f(s/s_\xi) \qquad s_\xi \sim (p-p_c)^{-1/\sigma}.$$

0305-4470/85/181149+07$02.25

In order to have an insight into the scaling behaviours on percolation we build up a regular model reflecting the above characteristic features of the process. In addition, a regular model has the advantage that we can obtain explicit expressions for the quantities of interest. To mimic the geometric texture of the percolating network just above the percolation threshold, it is necessary that a regular model is self-similar (fractal) on smaller length scales than the connectedness length but becomes homogeneous on large length scales. A regular model was proposed to be possessed of this property but was a poor approximation for the correlation length exponent. The band structures of the vibration problem were studied by the regular model but the structure of finite clusters was not examined (Nagatani 1985).

Now we try to imitate bond percolation with the help of a regular construction. The regular model is constructed by the following bond deletions. Bonds on the square lattice are recursively deleted via two rules. First, we apply the first rule of the bond deletion. Three construction stages of our regular model are shown in figure 1. The crosses, triangles and empty squares represent, respectively, bonds deleted at the first stage, the second stage and the third stage. Figure 2 represents a part of bonds deleted at the Nth stage according to the first rule. The bonds, indicated by crosses in figure 2, are deleted at the Nth stage from bonds connecting at the sites (i, j) which satisfy the relations:

$$\begin{aligned}
&\cos(2\pi i/3^N + 4\pi/3) = \cos(2\pi j/3^N + 4\pi/3) = 1,\\
&\cos(2\pi i/3^N + 2\pi/3) = \cos(2\pi j/3^N + 2\pi/3) = 1,\\
&\cos(2\pi i/3^N + 2\pi/3) = \cos(2\pi j/3^N + 4\pi/3) = 1\\
&\cos(2\pi i/3^N + 4\pi/3) = \cos(2\pi j/3^N + 2\pi/3) = 1.
\end{aligned} \qquad (1)$$

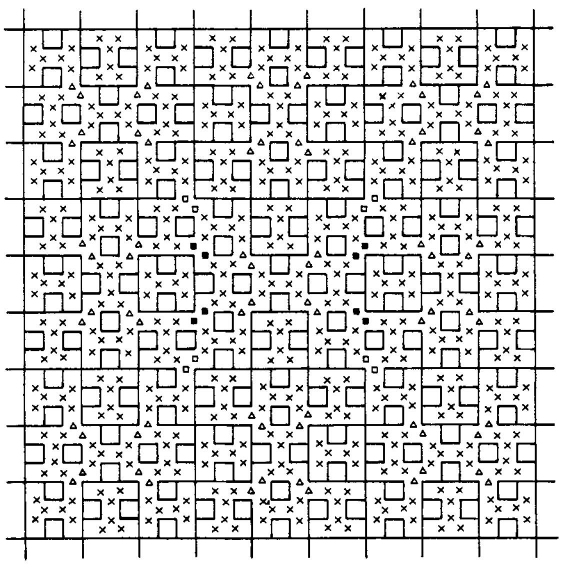

Figure 1. Three construction stages of the regular model. Crosses, triangles and open squares denote, respectively, the bonds deleted at the first, second and third stages according to the first rule of bond deletion. The full squares represent the bonds deleted by the second rule of bond deletion at the third stage.

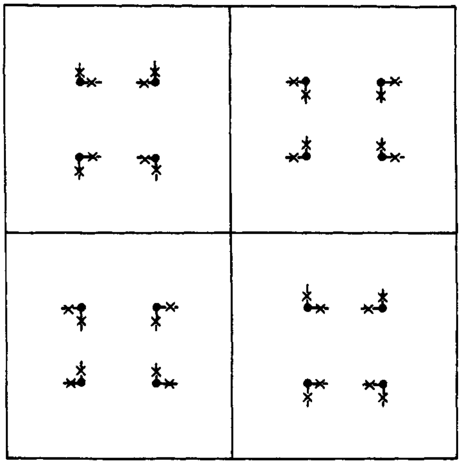

Figure 2. Bond deletions at the *N*th stage. The bonds indicated by crosses are deleted from bonds connecting at the sites (i, j) (represented by full circles) satisfying (1).

The system obtained at the N stages appears to be a superlattice made by nodes separated by a distance of $\xi = 3^N$, connected by quasi-linear links. Within this model, the correlation between two sites at distance $r < \xi$ is via a single link, but this link is a branching curve. The curve is identified as the Mandelbrot-Koch curve (Mandelbrot and Given 1984). We obtain the square lattice with self-similar structures on smaller length scales than the connectedness length $\xi = 3^N$. The concentration $c_1(N)$ of bonds, deleted at the Nth stage via the first rule of bond deletion, is given by

$$c_1(N) = 4/9^N \qquad (N \geqslant 2) \tag{2}$$

where $c_1(1) = \frac{3}{9}$. We obtain the percolating network (infinite cluster) by use of the first rule of bond deletion. Secondly, we apply the second rule of bond deletion to the resultant lattice. The second rule is applied to the islands separated from the percolating network. This rule works at stages larger than $N = 2$. The full squares in figure 1 represent bonds deleted at the third stage. Figure 3 shows the construction of finite clusters from a island separated from the percolating network at the fourth stage $(N = 4)$. Finite clusters in figure 4 are generated by use of the second rule of bond deletion. The finite clusters are found to be a fractal with the initiator of a square and the generator of the Mandelbrot-Koch curve. The second rule is summarised as follows: bonds into the islands separated from the percolating network are recursively deleted as the finite clusters shown in figure 4 generate. The concentration $c_2(n)$ of bonds, deleted at the Nth stage via the second rule of bond deletion, is given by

$$c_2(N) = 4 + 4c_2(N-1) \qquad (N \geqslant 4) \tag{3}$$

where $c_2(3) = 4$.

The concentration $p(N)$ of bonds after N stages is given by

$$p(N) = 1 - \sum_{n=1}^{N} c_1(n) - \sum_{n=3}^{N} c_2(n). \tag{4}$$

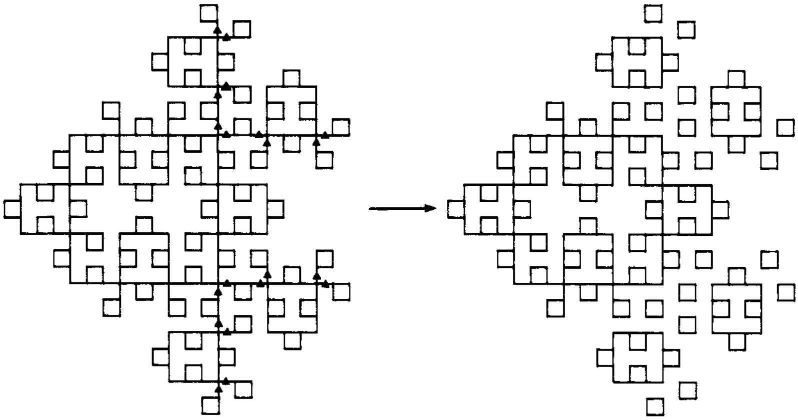

Figure 3. The construction of finite clusters from a island separated from the percolating network at the fourth stage ($N = 4$). An island is shown on the left-hand side. Bonds into the island indicated by the full triangles are deleted by the second rule. The finite clusters on the right-hand side generate.

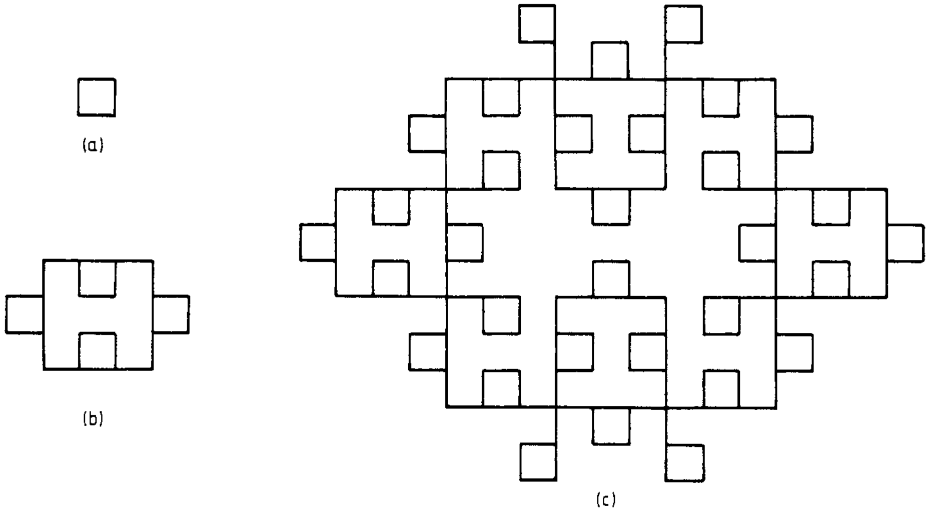

Figure 4. The finite clusters generated by the bond deletions. Finite clusters shown in figures 2(*a*), (*b*) and (*c*) represent, respectively, those generated at the second, third and fourth stages. A large cluster is the fractal with the initiator of square and the generator of the Mandelbrot–Koch curve.

When N is infinitely large, the concentration p approaches the critical value p_c:

$$p_c = \lim_{N\to\infty} p(N) = 1 - \tfrac{1}{3} - \tfrac{104}{2187}\left(\sum_{n=0}^{\infty} (\tfrac{1}{9})^n\right) - \tfrac{16}{2187}\left(\sum_{n=0}^{\infty} (\tfrac{4}{9})^n\right)$$
$$= \tfrac{3}{5}\ (= 0.6). \tag{5}$$

From (5) we obtain

$$\delta p (\equiv p(N) - p_c) \sim (\tfrac{4}{9})^N \sim \xi^{-2(1-\log 2/\log 3)}. \tag{6}$$

The connectedness length diverges as

$$\xi \sim (p-p_c)^{-\nu} \qquad \text{and} \qquad \nu = 0.5/(1-\log 2/\log 3)(=1.3547\ldots). \tag{7}$$

The value for the correlation length exponent agrees with that derived in a completely different fashion by Klein *et al* (1978) and was then thought to be perhaps exact. The most important feature of the regular model described above (figure 1) is that it is possible to get explicit expressions for the quantities characterising the approach towards the percolation threshold. The regular model is self-similar (fractal) on smaller length scales than the connectedness length, but becomes a homogeneous square lattice on large length scales. Our model is possessed of characteristic properties that the infinite cluster is composed of a backbone through which electrical current flows and dangling bonds hanging on it and the backbone consists of multiply connected 'blobs' joined by singly connected 'links'. The self-similar structure of the regular model is constructed by hierarchical extrapolation. The generator of the fractal is given by the Mandelbrot-Koch curve (Mandelbrot and Given 1984). The fractal dimension D of the infinite cluster and the fractal dimension D_b of its backbone are respectively given by $D=\log 8/\log 3$ and $D_b=\log 6/\log 3$. The exponent, describing the power law dependence on scale length L of the conductivity $L^{-t/\nu}$, is given by

$$t/\nu = \log R/\log b = \log(\tfrac{11}{4})/\log 3\ (=0.9207\ldots) \tag{8}$$

where we define R by assuming that for large n the two-point resistance of an order-n lattice of unit resistors is $\propto R^n$. By assuming the Einstein relation the spectral dimension d_s is given by $d_s = \log 64/\log 22$ ($=1.3454\ldots$) (Alexander and Orbach 1982, Alexander 1983).

The other important feature of the regular model described above (figure 1) is that it is possible to get explicit expressions for the quantities characterising the statistics of clusters defined in percolation. In order to obtain the cluster size distribution, one should note that the largest clusters generated in the kth stage of the process of bond deletions contains $s(k)\sim(8)^k$ bonds. For $s<(3^D)^N$ the cluster size distribution consists of a sum of delta functions:

$$n_s = \sum_{k=1}^{\infty} (\tfrac{1}{9})^k \left(\sum_{n=0}^{N-k} (\tfrac{4}{9})^n \right) \delta(s-(3^D)^k)\theta(1-s/(3^D)^N). \tag{9}$$

By spreading the delta functions over the interval we obtain

$$n_s \sim (\tfrac{1}{9})^k (1/3^D)^k \theta(1-s/(3^D)^N). \tag{10}$$

Taking into account that $s\sim(3^D)^k$, one can arrive at the scaling form

$$n_s \sim s^{-\tau}\theta(1-s/s_\xi) \tag{11}$$

with $\tau = 1+\log 9/\log 8 = 1+d/D$ where $s_\xi \sim (3^D)^N \sim (\xi)^D \sim (p-p_c)^{-\nu D}$, so $1/\sigma = \nu D$ is obtained. The scaling law $\beta/\nu = d-D$ is then satisfied. The threshold p_c obtained in (5) is somewhat higher than the exact value of 0.5. However, better agreement can be achieved by deleting further bonds of the lattice shown in figure 5. Bond deletions at lower stages (when N is small) do not change the scaling properties. The improved model obtained by the further bond deletions is shown in figure 5. The full triangles represent bonds deleted. The concentration of the bonds deleted is given by $c_3 = \frac{1}{9} - \frac{1}{81}$. We obtain the improved value $p_c = \frac{203}{405}$ ($=0.5012\ldots$) for the threshold. The scaling properties obtained above remain unchanged.

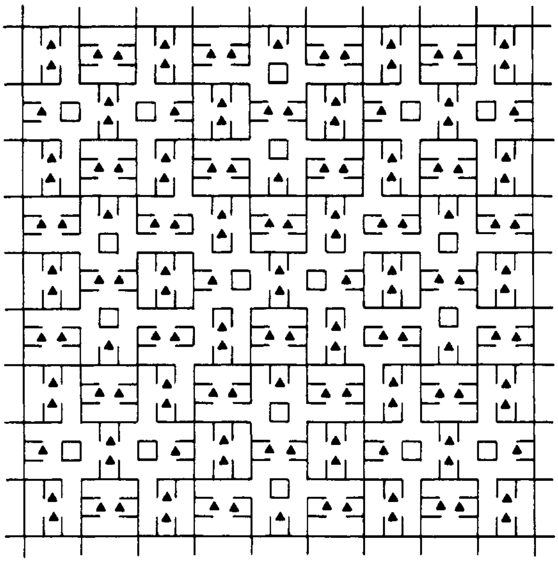

Figure 5. The improved model obtained by further bond deletions than shown in figure 1 in order to get better agreement of the threshold with the exact value. The full triangles indicate bonds further to those deleted in the model shown in figure 1.

Table 1 lists the geometric and physical properties, determined analytically by our regular model. In table 1, the second line shows estimated scaling exponents for the two-dimensional (random) percolation to compare more completely our results with those of random percolation. Conductivity and backbone exponents have only been numerically determined.

Table 1. List of the physical and geometric properties determined analytically by our regular model, compared with other sources: [a]Stauffer (1979, 1985); [b]Kapitulnik and Deutscher (1984); [c]Herrmann and Stanley (1984); [d]Hermann *et al* (1984); [e]Lobb and Frank (1984).

p_c	ν	τ	σ	D	D_b	t/ν
$\frac{203}{405}$	$\frac{0.5}{1-\log_3 2}$	$1+\log_8 9$	$\frac{1-\log_3 2}{0.5\log_3 8}$	$\log_3 8$	$\log_3 6$	$\log_3(\frac{11}{3})$
(0.5012)	(1.354)	(2.056)	(0.3899)	(1.892)	(1.630)	(0.9207)
0.5[a]	1.33[a]	2.05[a]	0.39[a]	1.90[b]	1.62[c]	0.97[d,e]

In summary, two-dimensional percolation can be imitated by the regular model. The regular construction of percolation, to simulate the scaling properties in the two-dimensional bond percolation, is shown to be possessed of characteristic properties of the infinite and finite clusters. Our critical exponents are very close to the exact ones. Due to the regular construction of percolation it is expected that the transport properties near the threshold are powerfully studied by the regular model.

I would like to thank the referee for his very helpful advice.

References

Aharony A, Gefen Y and Kantor Y 1984 *J. Stat. Phys.* **36** 795
Alexander S 1983 *Ann. Israel Phys. Soc.* **5** 149
Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L623
Ben-Avraham D and Havlin S 1983 *J. Phys. A: Math. Gen.* **16** L559
Coniglio A 1982 *J. Phys. A: Math. Gen.* **15** 3829
Deutscher G, Zallen R and Adler J 1983 *Percolation Structures and Processes. Ann. Israel Phys. Soc.* **5**
Given J A and Mandelbrot B B 1983 *J. Phys. A: Math. Gen.* **16** L565
Herrmann H J and Stanley H E 1984 *Phys. Rev. Lett.* **53** 1121
Herrmann H J, Derrida B and Vannimenus J 1984 *Phys. Rev.* B **30** 4080
Kapitulnik A and Deutscher G 1984 *J. Stat. Phys.* **36** 815
Kirkpatrick S 1979 *Les Houches Summer School on Ill-condensed matter* vol 32, ed R Balian, R Maynard and G Toulouse (Amsterdam: North-Holland) p 321
Klein W, Stanley H E, Reynolds P J and Coniglio A 1978 *Phys. Rev. Lett.* **41** 1145
Lobb C J and Frank D J 1984 *Phys. Rev.* B **30** 4090
Mandelbrot B B 1982 *The Fractal Geometry of Nature* (San Francisco: Freeman)
—— 1984a *J. Stat. Phys.* **34** 895
—— 1984b *J. Stat. Phys.* **36** 519
Mandelbrot B B and Given J A 1984 *Phys. Rev. Lett.* **52** 1853
Nagatani T 1985 *J. Phys. A: Math. Gen.* **18** L821
Stanley H E and Coniglio A 1983 *Ann. Israel Phys. Soc.* **5** 101
Stauffer D 1979 *Phys. Rep.* **54** 1
—— 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)
Vicsek T 1983 *J. Phys. A: Math. Gen.* **16** L647